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Preorderable topologies and order-representability of topological spaces

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We furnish characterizations of topologies that coincide with the lower topology or with the order topology of some total preorder defined on a set. Leaning on these characterizations we introduce some applications to the study of continuous and semicontinuous order-representability properties of topological spaces.

1. Introduction

Recent studies on the relationship between order and topology (see [14,5,9]) deal with total preorders that are continuous or semicontinuous with respect to a given topology. This means that the order topology (respectively, the upper or the lower topology) induced by the total preorder is coarser than the given topology.

In that framework, it would be helpful to identify topologies that coincide with the whole order topology or with the lower topology induced by some total preorder. These topologies are respectively called preorderable and lower preorderable.

We furnish characterizations of both classes of topologies.

The study of preorderable topologies may be applied to analyze the problem of the continuous representability property relative to total preorders defined on topological spaces. In this direction, we say that a topology defined on a nonempty set satisfies the continuous representability property (CRP) if every continuous total preorder defined on the given set admits a numerical representation by means of a continuous real-valued order-monomorphism. The property CRP can be used to characterize other topological properties of the given space. To put two recent examples we can say that it has already been used to analyze order-extension properties of topological spaces (see [9]), as well as to characterize various classical topological properties of a Banach space (see [10]), in Functional Analysis.

In a similar way, the study of lower preorderable topologies is related to the analysis of the semicontinuous representability property relative to total preorders defined on topological spaces. Now, a topology defined on a nonempty set satisfies

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the semicontinuous representability property (SRP) if every lower (upper) semicontinuous total preorder defined on the given set admits a numerical representation by means of a lower (upper) semicontinuous real-valued order-monomorphism. An important motivation to study this concept comes from Optimization Theory, in the search for extremal elements of suitable orderings defined on a set.

As a matter of fact, given a topological space (X,τ) the continuous (semicontinuous) representability property CRP (SRP) on (X,τ) is characterized by means of the satisfaction of the second countability axiom by every preorderable (lower preorderable) topology on X that is coarser than τ, as stated later in Section 5.

2. Preliminaries

A *preorder* $\precsim$ on an arbitrary nonempty set X is a binary relation on X which is reflexive and transitive. If $\precsim$ is a preorder on a set X, then we refer to the pair $(X,\precsim)$ as a preordered set.

An antisymmetric preorder is said to be an *order*. A *total preorder* $\precsim$ on a set X is a preorder such that if $x,y\in X$ then $[x\precsim y]$ or $[y\precsim x]$.

If $\precsim$ is a preorder on X, then we denote the associated *asymmetric* relation by $\prec$ and the associated *equivalence* relation by $\sim$ and these are defined, respectively, by $[x\prec y\Leftrightarrow (x\precsim y)\wedge\neg(y\precsim x)]$ and $[x\sim y\Leftrightarrow (x\precsim y)\wedge(y\precsim x)]$. Also, the *dual* preorder $\precsim_d$ is defined by $[x\precsim_d y\Leftrightarrow y\precsim x]$.

Let $(X,\precsim)$ be a totally preordered set and let $X/\sim$ be the set of equivalence classes. If $x\in X$ we denote the equivalence class of x by $[x]$. The preorder $\precsim$ on X induces a total order $\preccurlyeq$ on $X/\sim$ defined by $[x]\preccurlyeq[y]\Leftrightarrow x\precsim y$. Let $[x],[y]$ be two equivalence classes in $X/\sim$. Then we say that the ordered pair $([x],[y])\in(X/\sim)\times(X/\sim)$ is a *jump* if there is no $[z]\in X/\sim$ such that $[x]<[z]<[y]$, where $<$ denotes the asymmetric part of $\preccurlyeq$.

A subset Z of X is said to be *order-dense* in X with respect to $\precsim$, if $x,y\in X$ and $x\prec y$ imply that there exists $z\in Z$ such that $x\precsim z\precsim y$. $(X,\precsim)$ is said to be *order-separable* if it has a countable order-dense subset.

If $(X,\precsim)$ is a preordered set then a real-valued function $u:X\to\mathbb{R}$ is said to be:

(i) *increasing* if for every $x,y\in X$, $[x\precsim y\Rightarrow u(x)\leqslant u(y)]$;
(ii) *order-preserving* if f is increasing and $[x\prec y\Rightarrow u(x)<u(y)]$.

An order-preserving function is also said to be an *order-monomorphism*. A bijective order-monomorphism is an *order-isomorphism*.

If a nonempty set X is endowed with a topology τ then the total preorder $\precsim$ on X is said to be *continuously* (*lower, upper semicontinuously*) *representable* if there exists an order-monomorphism that is continuous (lower, upper semicontinuous) with respect to the topology τ on X and the usual topology on the real line $\mathbb{R}$.

Let $(X,\precsim)$ be a totally preordered set. The family of all sets of the form $L(x)=\{a\in X:\ a\prec x\}$ and $G(x)=\{a\in X:\ x\prec a\}$, where $x\in X$ is a subbasis for a topology $\tau_{\precsim}$ on X. The pair $(X,\tau_{\precsim})$ is called the *order topology* on X. The pair $(X,\tau_{\precsim})$ is called a *preordered topological space*.

Observe that the order topology $\tau_{\precsim}$ is the intersection of two topologies, namely the *lower topology* $\tau^l_{\precsim}$, a subbasis for which is given by the family of all sets of the form $G(x)=\{a\in X:\ x\prec a\}$ $(x\in X)$ and the *upper topology* $\tau^u_{\precsim}$, a subbasis for which is the family of all sets $L(x)=\{a\in X:\ a\prec x\}$ $(x\in X)$. Notice also that considering the dual order $\precsim_d$ it follows that the order topologies $\tau_{\precsim}$ and $\tau_{\precsim_d}$ coincide. Moreover the lower topology $\tau^l_{\precsim}$ coincides with the upper topology $\tau^u_{\precsim_d}$ and the upper topology $\tau^u_{\precsim}$ coincides with the lower topology $\tau^l_{\precsim_d}$.

If $(X,\precsim)$ is a preordered set and τ is a topology on X, then the preorder $\precsim$ is said to be τ-continuous on X if for each $x\in X$ the sets $\{a\in X:\ x\precsim a\}$ and $\{b\in X:\ b\precsim x\}$) are τ-closed in X.

In addition, the preorder $\precsim$ is said to be τ-lower (respectively, τ-upper) semicontinuous on X if for each $x\in X$ the set $\{a\in X:\ a\precsim x\}$ (respectively, the set $\{b\in X:\ x\precsim b\}$) is τ-closed in X.

Given a nonempty set X endowed with a topology τ, the topology τ on X is said to have the *continuous representability property* (CRP) if every continuous total preorder $\precsim$ defined on X admits a representation by means of a continuous order-monomorphism. (These topologies were studied in [14,9,11].)

Also, the topology τ on X is said to have the *semicontinuous representability property* (SRP) if every semicontinuous total preorder $\precsim$ defined on X admits a representation by means of a semicontinuous order-monomorphism (of the same type of semicontinuity). Topologies satisfying SRP were studied in [5,9].

Given a topological space (X,τ), the topology τ is said to be *preorderable* (respectively, *lower preorderable*) if it is the order topology $\tau_{\precsim}$ (respectively, the lower topology $\tau^l_{\precsim}$) of some *total preorder* $\precsim$ defined on X.

3. Preorderability of topologies

In this section we characterize preorderable and lower preorderable topologies, completing the panorama on *orderability of topologies* (see [18,17]).

Theorem 3.1. *Let (X,τ) be a topological space.*

(i) *The topology τ is lower preorderable if and only if it has a basis $\mathcal{B} = \{O_\alpha \subseteq X\colon \alpha \in A\}$, satisfying the following two conditions*:
   (a) *For every $\alpha, \beta \in A$ it holds that $O_\alpha \subseteq O_\beta$ or $O_\beta \subseteq O_\alpha$.*
   (b) *For every $\alpha \in A$ it holds that $\bigcap_{\gamma \in A,\, O_\alpha \subsetneq O_\gamma}(O_\gamma \setminus O_\alpha) \neq \emptyset$.*

(ii) *The topology τ is preorderable if and only if it has a subbasis $\mathcal{S} = \{O_\alpha \subseteq X\colon \alpha \in A\} \cup \{P_x\colon x \in X\}$, satisfying the following three conditions*:
   (a) *For every $\alpha, \beta \in A$ it holds that $O_\alpha \subseteq O_\beta$ or $O_\beta \subseteq O_\alpha$.*
   (b) *For every $\alpha \in A$ it holds that $\bigcap_{\gamma \in A,\, O_\alpha \subsetneq O_\gamma}(O_\gamma \setminus O_\alpha) \neq \emptyset$.*
   (c) *For every $x \in X$ we have that $P_x = \bigcup_{\alpha \in A,\, x \in O_\alpha}(X \setminus O_\alpha)$.*

(*A denotes a nonempty set of indexes.*)

Proof. (i) Assume first the existence of a basis of τ satisfying conditions (a) and (b) of the statement. Let us define a binary relation $\precsim$ on X by declaring $x \precsim y \Leftrightarrow (x \in O_\alpha \Rightarrow y \in O_\alpha)$, for every $\alpha \in A$ $(x, y \in X)$. It is straightforward to see that $\precsim$ is a total preorder. For any $x \in X$, we may observe that $G(x) = \bigcup_{\alpha \in A,\, x \notin O_\alpha} O_\alpha$. Hence $G(x) \in \tau$ $(x \in X)$. Thus the preorder $\precsim$ is τ-lower semicontinuous, or, in other words, the lower topology $\tau^l_\precsim$ satisfies that $\tau^l_\precsim \subseteq \tau$. As a matter of fact, these topologies coincide: To see this, we may observe that for every $\alpha \in A$, by condition (b) there exists an element $x \in \bigcap_{\gamma \in A,\, O_\alpha \subsetneq O_\gamma}(O_\gamma \setminus O_\alpha)$, and $G(x) = O_\alpha$. Hence $\tau \subseteq \tau^l_\precsim$. Therefore $\tau^l_\precsim = \tau$, so that the topology τ is lower preorderable.

For the converse, suppose that τ is lower preorderable. Let $\precsim$ be a total preorder on X such that τ coincides with $\tau^l_\precsim$. For every $x \in X$ we observe that $G(x)$ is τ-open by hypothesis. Moreover, $x \in \bigcup_{z \in X,\, G(z) \subsetneq G(x)}(G(z) \setminus G(x))$. In particular, $\bigcup_{z \in X,\, G(z) \subsetneq G(x)}(G(z) \setminus G(x)) \neq \emptyset$. Henceforth, the family $\{X\} \cup \{G(x)\colon x \in X\}$ satisfies the conditions (a) and (b) of the statement, and it is obviously a basis of the topology $\tau^l_\precsim = \tau$.

(ii) Suppose now that there is a subbasis of τ satisfying conditions (a) to (c) of the statement. As before, define the relation $\precsim$ on X by declaring $x \precsim y \Leftrightarrow (x \in O_\alpha \Rightarrow y \in O_\alpha)$, for every $\alpha \in A$ $(x, y \in X)$. Again, $\precsim$ is a total preorder such that, for any $x \in X$, $G(x) = \bigcup_{\alpha \in A,\, x \notin O_\alpha} O_\alpha$. Thus $G(x) \in \tau$ $(x \in X)$. But now we may also observe that for any $x \in X$, the set $L(x)$ is given by $L(x) = X \setminus \bigcap_{\alpha \in A,\, x \in O_\alpha} O_\alpha = \bigcup_{\alpha \in A,\, x \in O_\alpha}(X \setminus O_\alpha) = P_x$. As in part (i) we have again that for every $\alpha \in A$, we have $O_\alpha = G(x)$ for any $x \in \bigcap_{\gamma \in A,\, O_\alpha \subsetneq O_\gamma}(O_\gamma \setminus O_\alpha)$. It is clear now that the order topology $\tau_\precsim$ coincides with τ.

For the converse, suppose that τ is preorderable. Let $\precsim$ be a total preorder on X such that τ coincides with the order topology $\tau_\precsim$. As in the proof of part (i) of this Theorem 5.1, we have again that the family $\{X\} \cup \{G(x)\colon x \in X\}$ satisfies the conditions (a) and (b). Finally, we may observe that given $x \in X$, the set $L(x)$ is given by $L(x) = \bigcup_{y \in X,\, x \in G(y)}(X \setminus G(y))$. Therefore the family $\{X\} \cup \{G(x)\colon x \in X\} \cup \{L(x)\colon x \in X\}$ (that is actually a subbasis of $\tau = \tau_\precsim$) satisfies conditions (a) to (c) of the statement. $\square$

4. Properties of order topologies

Before studying continuous and semicontinuous order-representability properties (namely, CRP and SRP) on a topological space it seems necessary to start with a totally preordered set $(X, \precsim)$ endowed with its corresponding order topology $\tau_\precsim$, in order to understand better what else must happen to guarantee its representability through an order-monomorphism.

To start with, we quote a key result (see [7, Theorems 1.6.11 and 3.2.9]).

Lemma 4.1. *Let X be a nonempty set endowed with a total preorder $\precsim$. Let $\tau_\precsim$ be the order topology on X. Then the following conditions are equivalent*:

(i) *There exists an order-monomorphism from $(X, \precsim)$ into the real line $\mathbb{R}$ endowed with the usual order $\leqslant$.*
(ii) *The total preorder $\precsim$ is continuously representable* (*considering the order topology $\tau_\precsim$ on X and the usual topology on $\mathbb{R}$*) *through an order-monomorphism.*
(iii) *The order topology $\tau_\precsim$ is second countable.*
(iv) *The totally preordered set $(X, \precsim)$ is order-separable.*

A key to study of semicontinuous representability of a total preorder $\precsim$ defined on a nonempty set X is the behavior of its associated lower and upper topologies $\tau^l_\precsim$ and $\tau^u_\precsim$. Let us introduce some helpful result in this direction.

Theorem 4.2. *Let X be a nonempty set endowed with a total preorder $\precsim$. Then the order topology $\tau_\precsim$ is second countable if and only if the lower topology $\tau^l_\precsim$ is second countable.*

Proof. Let $\mathcal{B} = \{B_i\colon i \in \mathbb{N}\}$ be a countable basis for the order topology $\tau_\precsim$. Suppose that there exist $a, b \in X$ such that $a \prec b$ and there is no $c \in X$ such that $a \prec c \prec b$ (in other words, $([a], [b])$ is a jump). Since $G(a)$ is $\tau_\precsim$-open, there exists at least one element $B_k \in \mathcal{B}$ such that $b \in B_k$ and $b \prec d$ for every $d \in B_k \setminus \{x \in X\colon x \sim b\}$. Since $\mathcal{B}$ is countable, we conclude that the family of possible jumps that $\precsim$ defines on X is, at most, countable. Let $\mathcal{J} = \{([a_i], [b_i])\}_{i\in\mathbb{N}}$ be the family of all the jumps. (That is, $a_i, b_i \in X$; $a_i \prec b_i$; there is no $c \in X$ such that $a_i \prec c \prec b_i$ $(i \in \mathbb{N})$.) For any jump $([a_k], [b_k]) \in \mathcal{J}$, set $C_k = G(a_k)$ and $D_k = G(b_k)$. (Observe that D_k could eventually be empty if b_k is maximal in X with respect to $\precsim$.) Given now $B_k \in \mathcal{B}$, let $A_k = \bigcup_{x\in B_k} G(x)$. It is obvious that each A_k is an open set for the lower topology $\tau^l_{\precsim}$. (By the way, A_k might eventually be empty if X has a maximal element $s \in X$ and there is a last jump $([r], [s])$; $r \prec s$ in X with respect to $\precsim$.) Consider now the family $\mathcal{G}$ of $\tau^l_{\precsim}$-open subsets given by: $\mathcal{G} = \{A_i\colon\ A_i \neq \emptyset,\ i \in \mathbb{N}\} \cup \{C_i\colon\ i \in \mathbb{N}\} \cup \{D_i\colon\ D_i \neq \emptyset,\ i \in \mathbb{N}\}$. Let us see that $\mathcal{G}$ constitutes a (countable) basis of the topology $\tau^l_{\precsim}$: To see this, let $x \in X$ be such that $G(x) \neq \emptyset$ (so that $G(x)$ is a basic open set of $\tau^l_{\precsim}$). Let $y \in G(x)$. If $([x], [y])$ is a jump in X then $G(x) \in \mathcal{G}$ by construction. If, otherwise, there exists $z \in X$ such that $x \prec z \prec y$ then since $G(x)$ is also τ-open, and $\mathcal{B}$ is a basis of $\tau_{\precsim}$, there exists an element $B_k \in \mathcal{B}$ such that $z \in B_k$ and consequently $y \in A_k \subset G(x)$ and $A_k \in \mathcal{G}$. Thus we conclude that $\mathcal{G}$ is indeed a countable basis for the lower topology $\tau^l_{\precsim}$. Therefore the lower topology $\tau^l_{\precsim}$ is second countable.

For the converse, suppose that the lower topology $\tau^l_{\precsim}$ is second countable. Let $\mathcal{G} = \{G_n\colon\ n \in \mathbb{N}\}$ be a countable basis for the lower topology $\tau^l_{\precsim}$. Observe that each element $G_n \in \mathcal{G}$ satisfies that for every $y \in G_n$ and every $z \in X$ with $x \prec z$ it holds that $z \in G_n$. Consequently, if $n, k \in \mathbb{N}$ and $n \neq k$ it follows that $G_n \subsetneq G_k$ or $G_k \subsetneq G_n$. Thus, for every $n, k \in \mathbb{N}$ with $n \neq k$ we can select an element $x_{n_k} \in (G_n \setminus G_k) \cup (G_k \setminus G_n)$. It is plain that the set $\mathcal{X} = \{x_{n_k}\colon\ n, k \in \mathbb{N},\ n \neq k\}$ is countable. Suppose now that there exist $a, b \in X$ such that $a \prec b$ and $([a], [b])$ is a jump in X. Since $G(a)$ is $\tau^l_{\precsim}$-open, there exists at least one element $G_k \in \mathcal{B}$ such that $b \in G_k$ and $b \prec d$ for every $d \in G_k \setminus \{x \in X\colon\ x \sim b\}$ (i.e., b is a minimal element of G_k as regards $\precsim$). Since $\mathcal{G}$ is a countable basis for the lower topology $\tau^l_{\precsim}$, it follows that the family of possible jumps that $\precsim$ defines on X is, at most, countable. Let $\mathcal{J} = \{([a_i], [b_i])\}_{i\in\mathbb{N}}$ be the family of all the possible jumps. For a given jump $([a_k], [b_k]) \in X$ we select two elements u_k and v_k such that $u_k \in [a_k]$ and $v_k \in [b_k]$. Let $\mathcal{Y} = \{u_i, v_i\colon\ ([a_i], [b_i])$ is a jump in X with respect to $\precsim$ $(i \in \mathbb{N})\}$. Let $\mathcal{Z} = \mathcal{X} \cup \mathcal{Y}$. Observe that $\mathcal{Z}$ is countable, by definition. Let now $a, b \in X$ be such that $a \prec b$ and $([a], [b])$ is not a jump. Take an element $c \in X$ such that $a \prec c \prec b$. If either $([a], [c])$ or $([c], [b])$ is a jump, then there exists $c' \in [c]$ such that $c' \in \mathcal{Y}$. If there exist $p, q \in X$ such that $a \prec p \prec c \prec q \prec b$, then since $G(a)$ and $G(p)$ are both $\tau^l_{\precsim}$-open and $\mathcal{G}$ is a basis for $\tau^l_{\precsim}$, set $G(a) = \bigcup_{n\in\mathbb{N}} G_{r_n}$ and $G(p) = \bigcup_{n\in\mathbb{N}} G_{s_n}$. Since $p \in G(a) \setminus G(p)$, there exists $k \in \mathbb{N}$ such that $p \in G_{r_k}$. Hence $G(p) \subseteq G_{r_k}$. Because $c \in G(p) \setminus G(c)$, there exists $n \in \mathbb{N}$ such that $c \in G_{s_n}$. Hence $G(c) \subseteq G_{s_n}$. By construction, the element $x_{r_k, s_n} \in \mathcal{X}$ lies also in $G(r_k) \setminus G(s_n) \subset G(a) \setminus G(c)$. Thus $a \prec x_{r_k, s_n} \precsim c \prec b$. Therefore, for every $a, b \in X$ such that $a \prec b$ we may always find an element $z \in \mathcal{Z}$ such that $a \precsim z \precsim b$. This implies that the totally ordered set $(X, \precsim)$ is order-separable and consequently the order topology $\tau_{\precsim}$ is second countable by Lemma 4.1. $\square$

5. Continuous and semicontinuous representability properties on topological spaces

In this section we furnish characterizations of topologies that satisfy the continuous (semicontinuous) representability property CRP (SRP).

Theorem 5.1. *Let (X, τ) be a topological space. The topology τ satisfies CRP (SRP) if and only if all its preorderable (lower preorderable) subtopologies are second countable. In particular, SRP implies CRP.*

Proof. (i) If τ' is a preorderable subtopology of τ, the total preorder $\precsim$ that defines $\tau' = \tau_{\precsim}$ is obviously τ-continuous. Conversely, if $\precsim$ is a τ-continuous preorder on X, then its order topology $\tau_{\precsim}$ is a preorderable subtopology of τ. The definition of the property CRP says that τ satisfies CRP when every τ-continuous total preorder $\precsim$ defined on X is continuously representable (considering on X the order topology $\tau_{\precsim}$ and on $\mathbb{R}$ the usual Euclidean topology). By Lemma 4.1 this is equivalent to say that for every τ-continuous total preorder $\precsim$ defined on X, the order topology $\tau_{\precsim}$ is second countable. The result in the statement follows now immediately from the bijection between "order topologies of τ-continuous total preorders" and "preorderable subtopologies of τ".

(ii) If τ' is a lower preorderable subtopology of τ, the total preorder $\precsim$ that defines $\tau' = \tau_{\precsim}$ is obviously τ-lower semicontinuous. Moreover, its dual $\precsim_d$ is τ-upper semicontinuous, and, in addition, $\tau_{\precsim}$ and $\tau_{\precsim_d}$ coincide. Conversely, if $\precsim$ is a τ-lower semicontinuous preorder on X, then its order topology $\tau_{\precsim}$ is a lower preorderable subtopology of τ. Also, if $\precsim$ is a τ-upper semicontinuous preorder on X, then its associated dual preorder $\precsim_d$ is actually τ-lower semicontinuous, and its order topology $\tau_{\precsim_d}$, that coincides with the topology $\tau_{\precsim}$, is a lower preorderable subtopology of τ. Thus we see that there is a bijection between τ-semicontinuous preorders (of any type) defined on X and lower preorderable subtopologies of τ.

Suppose now that the topology τ has SRP. This means that every semicontinuous total preorder $\precsim$ defined on X admits a representation by means of a semicontinuous order-monomorphism (of the same type of semicontinuity). In particular, every semicontinuous total preorder $\precsim$ is representable through an order-monomorphism, so that the topology $\tau_{\precsim}$ is second countable by Lemma 4.1. Hence the topologies $\tau^l_{\precsim}$ and $\tau^u_{\precsim} = \tau^l_{\precsim_d}$ are also second countable by Theorem 4.2. Due to the aforementioned bijection, this implies that all the lower preorderable subtopologies of τ are second countable.

Conversely, let us assume now that all the lower preorderable subtopologies of τ are second countable. Suppose that $\precsim$ is a lower semicontinuous preorder defined on X. This means that $\tau^l_{\precsim}$ is coarser than τ. By hypothesis, $\tau^l_{\precsim}$ is second countable. By Theorem 4.2 this implies that $\tau_{\precsim}$ is also second countable. By Lemma 4.1, it follows that $\precsim$ is representable by an order-monomorphism $f : X \to \mathbb{R}$ that is continuous if we consider on X the *order* topology $\tau_{\precsim}$ and on $\mathbb{R}$ the usual topology. Given $a \in \mathbb{R}$ we have that $f^{-1}(a, +\infty) = \bigcup_{\{x \in X\colon\ f(x) \geqslant a\}} G(x)$ is an open set in $\tau^l_{\precsim}$, hence in τ. Therefore f is lower semicontinuous (now considering the topology τ on X and the usual topology on $\mathbb{R}$). Finally, if $\precsim$ is an upper semicontinuous preorder defined on X, its dual $\precsim_d$ is a lower semicontinuous preorder defined on X. As before, $\precsim_d$ will be representable by an order-monomorphism g that is lower semicontinuous (again with respect to the topology τ on X and the Euclidean topology on $\mathbb{R}$). This plainly implies that $-g$ is an upper semicontinuous monomorphism that represents $\precsim$. $\square$

Remark 5.2. In general CRP does not imply SRP (see [5,9]).

Using the characterizations of preorderable and lower preorderable topologies introduced in Theorem 3.1, we can obtain now further characterizations of topologies satisfying CRP or SRP. To do so, first we introduce some definitions.

Definition. Let (X, τ) be a topological space. Let $\mathcal{S} = \{O_\alpha \subseteq X\colon\ \alpha \in A\} \cup \{P_x\colon\ x \in X\}$ be a family of subsets of X. (A denotes a nonempty set of indexes.) The family $\mathcal{S}$ is said to be *continuously preorder-generating* if it satisfies the following conditions:

(a) The elements O_α and P_x ($\alpha \in A,\ x \in X$) are all τ-open.
(b) For every $\alpha, \beta \in A$ it holds that $O_\alpha \subseteq O_\beta$ or $O_\beta \subseteq O_\alpha$.
(c) For every $\alpha \in A$ it holds that $\bigcap_{\gamma \in A,\ O_\alpha \subsetneq O_\gamma} (O_\gamma \setminus O_\alpha) \neq \emptyset$.
(d) For every $x \in X$ we have that $P_x = \bigcup_{\alpha \in A,\ x \in O_\alpha} (X \setminus O_\alpha)$.

Definition. Let (X, τ) be a topological space. Let $\mathcal{F} = \{O_\alpha \subseteq X\colon\ \alpha \in A\}$ be a family of subsets of X. (A denotes a nonempty set of indexes.) The family $\mathcal{F}$ is said to be *semicontinuously preorder-generating* if it satisfies the following conditions:

(a) Each O_α is a τ-open set.
(b) For every $\alpha, \beta \in A$ it holds that $O_\alpha \subseteq O_\beta$ or $O_\beta \subseteq O_\alpha$.
(c) For every $\alpha \in A$ it holds that $\bigcap_{\gamma \in A,\ O_\alpha \subsetneq O_\gamma} (O_\gamma \setminus O_\alpha) \neq \emptyset$.

After a glance to the proof of Theorem 3.1, we observe now that given a topological space (X, τ) it holds that each continuously preorder-generating family $\mathcal{S}$ defines a total preorder $\precsim_{\mathcal{S}}$ on X by declaring that $x \precsim_{\mathcal{S}} y \Leftrightarrow (x \in O_\alpha \Rightarrow y \in O_\alpha)$, for every $\alpha \in A$ ($x, y \in X$). In addition, the order topology $\tau_{\precsim_{\mathcal{S}}}$ associated to $\precsim_{\mathcal{S}}$ is coarser than τ. In other words, the total preorder $\precsim_{\mathcal{S}}$ is τ-continuous. Obviously, the topology $\tau_{\precsim_{\mathcal{S}}}$ is preorderable.

In the same way, each semicontinuously preorder-generating family $\mathcal{F}$ on a topological space (X, τ) also defines a total preorder $\precsim_{\mathcal{F}}$ on X by declaring that $x \precsim_{\mathcal{F}} y \Leftrightarrow (x \in O_\alpha \Rightarrow y \in O_\alpha)$, for every $\alpha \in A$ $(x, y \in X)$. In this case, the lower topology $\tau^l_{\precsim_{\mathcal{F}}}$ is coarser than τ. In other words, the total preorder $\precsim_{\mathcal{F}}$ is τ-lower semicontinuous. Needless to say that the topology $\tau^l_{\precsim_{\mathcal{F}}}$ is lower preorderable.

Thus we conclude that, given a topological space (X, τ), to have a preorderable (respectively, lower preorderable) coarser topology is actually equivalent to define a continuously (respectively, semicontinuously) preorder-generating family.

This easy fact allows us now to give characterizations of CRP and SRP on a topological space (X, τ), *working directly with continuously and semicontinuously preorder-generating families*. First we introduce a definition.

Definition. Let (X, τ) be a topological space. Let $\mathcal{T} = \{T_\alpha \subseteq X\colon\ \alpha \in A\}$ be a family of subsets of X, where A stands for a set of indexes. Let $x \in X$. The set $T^*_x = \bigcup_{\alpha \in A,\ x \notin T_\alpha} T_\alpha$ is said to be the *vanishing* set of x with respect to the family $\mathcal{T}$. (Observe that T^*_x may eventually be empty.)

Theorem 5.3. *Let (X, τ) be a topological space.*

(i) *The topological space (X, τ) satisfies the continuous representability property CRP if and only if for every continuously preorder-generating family $\mathcal{S} = \{O_\alpha \subseteq X\colon\ \alpha \in A\} \cup \{P_x\colon\ x \in X\}$ (where A is a set of indexes), there exists a countable subset $\{x_n\colon\ n \in \mathbb{N}\} \subseteq X$ of elements of X such that for every $\alpha, \beta \in A$ with $O_\alpha \subsetneq O_\beta$ there exists $k \in \mathbb{N}$ such that $O_\alpha \subseteq O^*_{x_k} \subseteq O_\beta$, where $O^*_{x_k}$ stands for the vanishing set of the element x_k with respect to the subfamily $\mathcal{O} = \{O_\alpha \subseteq X\colon\ \alpha \in A\}$ of $\mathcal{S}$.*
(ii) *The topological space (X, τ) satisfies the semicontinuous representability property SRP if and only if for every semicontinuously preorder-generating family $\mathcal{F} = \{O_\alpha \subseteq X\colon\ \alpha \in A\}$ (where A is a set of indexes), there exists a countable subset $\{x_n\colon\ n \in \mathbb{N}\} \subseteq X$ of elements of X such that for every $\alpha, \beta \in A$ with $O_\alpha \subsetneq O_\beta$ there exists $k \in \mathbb{N}$ such that $O_\alpha \subseteq O^*_{x_k} \subseteq O_\beta$, where $O^*_{x_k}$ stands for the vanishing set of the element x_k with respect to $\mathcal{F}$.*

Proof. (i) Suppose first that the topological space (X,τ) satisfies the continuous representability property CRP. Let $\mathcal{S} = \{O_\alpha \subseteq X\colon \alpha \in A\} \cup \{P_x\colon x \in X\}$ be a continuously preorder-generating family on (X,τ). By CRP, the total preorder $\precsim_{\mathcal{S}}$ associated to $\mathcal{S}$ is continuously representable (considering the order topology $\tau_{\precsim_{\mathcal{S}}}$ on X and the usual topology on $\mathbb{R}$). By Lemma 4.1, there exists a countable order-dense subset $\{x_n\colon n \in \mathbb{N}\} \subseteq X$. Suppose now that $O_\alpha \subsetneq O_\beta$. By the proof of Theorem 3.1, there exist $y_\alpha, y_\beta \in X$ such that $O_\alpha = G(y_\alpha)$, $O_\beta = G(y_\beta)$ and $y_\beta \prec_{\mathcal{S}} y_\alpha$. Therefore, there exists $k \in \mathbb{N}$ such that $y_\beta \precsim_{\mathcal{S}} x_k \precsim_{\mathcal{S}} y_\alpha$. This implies that $G(y_\alpha) \subseteq G(x_k) \subseteq G(y_\beta)$. Thus $O_\alpha \subseteq O^*_{x_k} \subseteq O_\beta$, by construction.

Conversely, assume that for every continuously preorder-generating family $\mathcal{S} = \{O_\alpha \subseteq X\colon \alpha \in A\} \cup \{P_x\colon x \in X\}$ there exists a countable subset $\{x_n\colon n \in \mathbb{N}\} \subseteq X$ such that for every $\alpha, \beta \in A$ with $O_\alpha \subsetneq O_\beta$ there exists $k \in \mathbb{N}$ such that $O_\alpha \subseteq O^*_{x_k} \subseteq O_\beta$. Let $\precsim$ be a τ-continuous total preorder. It is plain that the family $\mathcal{S} = \{O_x = G(x)\colon x \in X\} \cup \{P_x = L(x)\colon x \in X\}$ is continuously preorder-generating. Calling $\mathcal{O} = \{O_x\colon x \in X\}$, by hypothesis there exists a countable subset $\{x_n\colon n \in \mathbb{N}\} \subseteq X$ such that for every $z, t \in X$ with $O_z \subsetneq O_t$ there exists $k \in \mathbb{N}$ such that $O_z \subseteq O^*_{x_k} \subseteq O_t$. Notice that $O_z = G(z)$ and $O_t = G(t)$ so that $O_z \subsetneq O_t \Leftrightarrow t \prec z$, and $O_z \subseteq O^*_{x_k} \subseteq O_t$ actually implies that $t \precsim x_k \precsim z$. Therefore $\{x_n\colon n \in \mathbb{N}\} \subseteq X$ is a countable order-dense subset of X. Hence by Lemma 4.1 the preorder $\precsim$ is continuously representable (considering the order topology $\tau_{\precsim}$ on X and the usual topology on $\mathbb{R}$). Since $\precsim$ is τ-continuous, we conclude that $\precsim$ is also continuously representable (but now considering the given topology τ on X and the usual topology on $\mathbb{R}$).

(ii) Now, assume first that the topological space (X,τ) satisfies the semicontinuous representability property SRP. Let $\mathcal{F} = \{O_\alpha \subseteq X\colon \alpha \in A\}$ be a semicontinuously preorder-generating family on (X,τ). By SRP, the total preorder $\precsim_{\mathcal{F}}$ associated to $\mathcal{F}$ is in particular representable. By Lemma 4.1 again, there exists a countable order-dense subset $\{x_n\colon n \in \mathbb{N}\} \subseteq X$. Suppose now that $O_\alpha \subsetneq O_\beta$. By the proof of Theorem 3.1, there exist $y_\alpha, y_\beta \in X$ such that $O_\alpha = G(y_\alpha)$, $O_\beta = G(y_\beta)$ and $y_\beta \prec_{\mathcal{S}} y_\alpha$. Therefore, there exists $k \in \mathbb{N}$ such that $y_\beta \precsim_{\mathcal{S}} x_k \precsim_{\mathcal{S}} y_\alpha$. This implies that $G(y_\alpha) \subseteq G(x_k) \subseteq G(y_\beta)$, so that $O_\alpha \subseteq O^*_{x_k} \subseteq O_\beta$.

Finally, assume now that for every semicontinuously preorder-generating family $\mathcal{F} = \{O_\alpha \subseteq X\colon \alpha \in A\}$ there exists a countable subset $\{x_n\colon n \in \mathbb{N}\} \subseteq X$ such that for every $\alpha, \beta \in A$ with $O_\alpha \subsetneq O_\beta$ there exists $k \in \mathbb{N}$ such that $O_\alpha \subseteq O^*_{x_k} \subseteq O_\beta$. Let $\precsim$ be a τ-lower semicontinuous total preorder. (The arguments for τ-upper semicontinuous total preorders are entirely analogous.) It is plain that the family $\mathcal{F} = \{O_x = G(x)\colon x \in X\}$ is semicontinuously preorder-generating. By hypothesis there exists a countable subset $\{x_n\colon n \in \mathbb{N}\} \subseteq X$ such that for every $z, t \in X$ with $O_z \subsetneq O_t$ there exists $k \in \mathbb{N}$ such that $G(z) = O_z \subseteq O^*_{x_k} \subseteq O_t = G(t)$. Hence it follows that $t \prec z$ and $t \precsim x_k \precsim z$. Therefore $\{x_n\colon n \in \mathbb{N}\} \subseteq X$ is a countable order-dense subset of X. Hence by Lemma 4.1 the preorder $\precsim$ is continuously representable (considering the order topology $\tau_{\precsim}$ on X and the usual topology on $\mathbb{R}$), and, since $\precsim$ is lower-semicontinuous, this implies that $\precsim$ is representable through a lower-semicontinuous utility function from the topological space (X,τ) to the real line $\mathbb{R}$ endowed with its usual Euclidean order and topology. $\square$

To conclude, we give an alternative proof of a classical result on CRP.

Corollary 5.4. *([12]) Let X be a topological space τ. If the topological space (X,τ) is connected and separable, then it satisfies the continuous representability property (CRP). The converse is not true in general.*

Proof. Since (X,τ) is separable, there exists a countable subset $\{x_n\colon n \in \mathbb{N}\} \subseteq X$ that is τ-dense. Let $\mathcal{S} = \{O_\alpha \subseteq X\colon \alpha \in A\} \cup \{P_x\colon x \in X\}$ be a continuously preorder-generating family on (X,τ). Let $\alpha, \beta \in A$ be such that $O_\alpha \subsetneq O_\beta$. Let $U = \bigcap_{\gamma \in A,\, O_\alpha \subsetneq O_\gamma} O_\gamma$. We observe that $X \setminus U = \bigcup_{\gamma \in A,\, O_\alpha \subsetneq O_\gamma} (X \setminus O_\gamma) = \bigcup_{x \in O_\alpha} (\bigcup_{\gamma \in A,\, x \in O_\gamma} (X \setminus O_\gamma)) = \bigcup_{x \in O_\alpha} P_x$. Therefore $X \setminus U$ is τ-open by definition of a continuously preorder-generating family.

Consequently, U is a τ-closed subset of X. In addition, U is nonempty because by hypothesis $\bigcap_{\gamma \in A,\, O_\alpha \subsetneq O_\gamma} (O_\gamma \setminus O_\alpha) \neq \emptyset$.

Let now $V = X \setminus O_\beta$. This subset $V \subseteq X$ is obviously τ-closed since $O_\beta \in \tau$. Moreover, V is nonempty because $\bigcap_{\gamma \in A,\, O_\beta \subsetneq O_\gamma} (O_\gamma \setminus O_\beta) \neq \emptyset$. Now we observe that $O_\alpha \subsetneq O_\beta \Rightarrow U \subseteq O_\alpha \subsetneq O_\beta \Rightarrow U \cap (X \setminus O_\beta) = \emptyset$. Thus we have $U \cap V = \emptyset$, by definition of V. Since (X,τ) is connected, $X \setminus (U \cup V)$ is a nonempty τ-open set. Consequently, there exists $k \in \mathbb{N}$ such that $x_k \in X \setminus (U \cup V)$. It is straightforward to see now that $O_\alpha \subseteq O^*_{x_k} \subseteq O_\beta$, where $O^*_{x_k}$ denotes the vanishing set of x_k with respect to the subfamily $\mathcal{O} = \{O_\alpha \subseteq X\colon \alpha \in A\} \subseteq \mathcal{S}$.

An easy example that shows that the converse is not true is a finite set X with more than one point, endowed with the discrete topology. $\square$

6. Final comments

In order to clarify the objectives and achievements of the present paper, in this section we compare some notions and results introduced here with other items already appeared in the literature. In the final part, we include some open problems and suggestions for further studies.

6.1. Discussion

Our main objective has been to study *preorderable topologies*, in order to complete the panorama of the classical results on *orderability of topologies*. (See e.g. [3,16,19,18], or, recently, [1]. For a historical account, consult [17].)

The classical studies on orderability of topologies mainly deal with topologies related to a *total* (linear) order, disregarding the study of topologies induced by total *preorders*. Although a total preorder $\precsim$ defined on a set X obviously defines a linear order on a quotient through the equivalence classes given by $\sim$, the problem here is that, a priori, *we only have a topology on a set*, but *not* a linear order, nor a total preorder.

Roughly speaking, topologies given by total preorders have quotients that correspond to orderable topologies. However (*and this is the difficulty*!) we cannot use (directly) the classical results on orderability to decide whether a given topology is induced by a total *preorder* or not, since we cannot guess (at this first stage) which could be the suitable quotient.

That is, we should identify which topologies are induced by total preorders, working directly on, say, the family of open subsets that define the topology. The total preorder involved would appear *a posteriori* as a by-product. *This is the work done in Section* 3 *of the present paper*.

Needless to say that this has various motivations, not only the *main* idea of filling the gap and *completing the panorama on orderability of topologies*. Thus, the consideration of topologies induced by total preorders is a key tool to analyze properties of continuous ordinal representability properties of topologies, *namely the properties CRP and SRP studied in Section* 5 *above*.

At this stage, it is important to say that several characterizations of the properties CRP and SRP have also been previously obtained in the literature, using different techniques. For instance, CRP was already considered and characterized in the pioneer work [13] (see also [14] where more characterizations were given), and SRP has also been characterized in [5].

But, in our opinion, it is still interesting to get *alternative* characterizations of CRP (SRP) based on the notions introduced here, that correspond to preorderability (lower preorderability) of topologies. This not only shows the strength and scope of those new concepts, but, in addition, furnishes a *unified approach* to deal with both CRP and SRP. Moreover, the main characterization (Theorem 5.1 above) is obtained by, so-to-say, "using only two ingredients", namely the *preorderability* (or lower preorderability) of subtopologies and the satisfaction of the *second countability axiom*. This also links with other classical studies about covering properties on ordered topological spaces (see [15]).

Concerning the techniques and concepts introduced, it is true that some of the concepts are similar, at first glance, to other ones that have already been used in the literature to characterize either CRP or SRP. Thus, for instance, the notion of a *continuously preorder-generating family* that we introduced in Section 5 appears to be similar to the concept of an $\mathcal{R}$*-separable system* which allows to recover a characterization of the continuous representability property (see [14] for details). It is important to point out that other different families of open sets, somewhat similar to $\mathcal{R}$-separable systems or continuously preorder-generating families have also been introduced in this literature to deal with continuous representations of total orders defined on topological spaces. This is the case, to put only an example, of *countable decreasing scales* (see [8,4]). However, *there are subtle nuances and differences between those concepts and the* (*alternative*) *results that they induce*, where we may encounter various characterizations of CRP, at least in particular situations. (See the comparisons and remarks that appear in [2,4].)

Our intention when we introduced that concept of a *continuously preorder-generating family* was *twofold*. On the one hand, we defined that notion *in order to describe which families of open sets of a given topology on a set give rise to a preorderable subtopology*. On the other hand (and implicitly understood as something that is done in a second step!) it is clear that we may use this fact jointly with Theorem 5.1 to obtain a new characterization of CRP, *as an application*. As a matter of fact, Theorem 5.3 has been introduced to, so-to-say, "directly read" the satisfaction of the second countability axiom by the topology generated by a continuously (semicontinuously) preorder-generating family *in terms of* (*only*) *the members of that family*.

It is instructive to compare Theorem 5.3(i) with other existing characterizations of CRP. In particular, given a topological space (X,τ), in Proposition 5.1 in [14] we find a characterization of CRP that is provided on the basis of controlling topological properties of *two* different kinds of families of subsets of X. By means of the first class of families, that consist of open–closed sets with some additional properties, we must control the so-called *open–closed countable chain condition* (*OCCC*). The second class of families involved in the statement of Proposition 5.1 in [14] corresponds to the so-called *linear separable systems* of X and we must control some type of separability condition. In part (i) of Theorem 5.3 we only need to control *one* kind of families, namely the *continuously preorder-generating* ones. Moreover, our families allow us to characterize the preorderability of subtopologies of (X,τ).

We may finally observe that the main characterization of CRP obtained in Proposition 5.1 in [14] also leans on the satisfaction of the second countability axiom by suitable subtopologies (namely, the so-called "linearly ordered" ones, where in that context *linearly ordered* means that the family of open sets of that topology is linearly ordered by set inclusion) of a given topology defined on a set, *as in our Theorem* 5.1 *above*. However, the subtopologies considered in Proposition 5.1 in [14] may or may not be preorderable. Also, as mentioned before, Proposition 5.1 in [14] uses an additional condition, namely the *open–closed countable chain condition* (*OCCC*), that we do not need here.

6.2. Open problems

Throughout the paper we have studied and analyzed topological spaces (X,τ) such that the topology τ is the order topology or the lower topology of some total preorder defined on X.

But nothing similar is known for topologies induced by other different kinds of binary relations (that may fail to be total preorders).

Thus, given an asymmetric binary relation $\prec$ on a nonempty set X, we can also consider the sets of the form $G(x) = \{a \in X\colon\ x \prec a\}$ $(x \in X)$ as well as the sets $L(x) = \{a \in X\colon\ a \prec x\}$ $(x \in X)$, and consequently define the lower topology, the upper topology, and their intersection topology (again called *order topology*) that $\prec$ induces on X.

However, it is *no longer* true in general that the binary relation $\precsim$ given by $a \precsim b \Leftrightarrow \neg(b \prec a)$ $(a, b \in X)$ is a total preorder. Important particular cases correspond to interval orders (see e.g. [7]).

We recall that an *interval order* $\prec$ is an asymmetric binary relation such that $[(x \prec y)$ and $(z \prec t)] \Rightarrow [(x \prec t)$ or $(z \prec y)]$ $(x, y, z, t \in X)$.

An interval order $\prec$ defined on X is called *representable* if there exist two real valued maps $u, v : X \to \mathbb{R}$ such that $x \prec y \Leftrightarrow v(x) < u(y)$ $(x, y \in X)$.

In this framework, *it is an open problem to characterize topologies that coincide with the lower topology or with the order topology of an interval order*.

In addition, we could say that a topological space (X, τ) satisfies the *continuous representability property for interval orders* (*CRP-I.O*) if for every interval order $\prec$ defined on X and such that all the sets $G(x) = \{a \in X\colon\ x \prec a\}$ $(x \in X)$ and $L(x) = \{a \in X\colon\ a \prec x\}$ $(x \in X)$ are τ-open, there exists a pair of continuous functions $u, v : X \to \mathbb{R}$ (where X is endowed with the topology τ and $\mathbb{R}$ with the usual topology), such that $x \prec y \Leftrightarrow v(x) < u(y)$ $(x, y \in X)$. Similarly, we could define the *semicontinuous representability property for interval orders* (*SRP-I.O*), see [6].

As far as we know, *the analysis and characterization of topological spaces that satisfy CRP-I.O or SRP-I.O also remains open*.

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